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ON 3-DIMENSIONAL ALMOST ALPHA KENMOTSU MANIFOLDS

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Abstract

This paper deals with the geometry of almost alpha Kenmotsu manifold satisfying some certain tensor conditions for 3-dimensional case. In particular, we study projectively and concircularly semi-symmetric conditions. Finally, we give an illustrative example on such manifold with dimension 3.

Keywords:

Kenmotsu manifold;
Almost alpha Kenmotsu manifold; Eta parallelity;
Semi-symmetry.

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1. Introduction

Manifolds known as Kenmotsu manifolds have been studied by Kenmotsu (see [6]). The author set up one of the three classes of almost contact Riemannian manifolds whose automorphism group attains the maximum dimension (see [14]). A Kenmotsu manifold can be defined as a normal almost contact metric manifold such that $d\eta = 0$ and $d\Phi = 2\eta \wedge \Phi$.

It is well known that Kenmotsu manifolds can be characterized through their Levi-Civita connection, by $(\nabla_X \varphi)Y = g(\varphi X, Y)\xi - \eta(Y)\varphi X$, for any vector fields X and Y. Kenmotsu defined a structure closely related to the warped product which was characterized by tensor equations. He proved that such a manifold M^{2n+1} is locally a warped product $(-\varepsilon, +\varepsilon) \times_f N^{2n}$ being a Kaehlerian manifold and $f(t) = ce^t$ where c is a positive constant. Moreover, Kenmotsu showed locally symmetric Kenmotsu manifolds are of constant curvature -1 that means locally symmetry is a strong restriction for Kenmotsu manifolds. Also, the author realized that if Kenmotsu structure satisfies the Nomizu's condition, i.e., $R \cdot R = 0$, then it has negative constant curvature and if Kenmotsu manifold is conformally flat, then the manifold is a space of constant negative curvature -1 for dimension greater than 3 (see [8]).

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The notion of semi-symmetric manifold is defined by

$$R(X,Y)\cdot R=0, \tag{1.1}$$

for all vector fields X,Y on M, where $R(X,Y)$ acts as a derivation on R (see [8]). Such a space is called "semi-symmetric space" since the curvature tensor of (M,g) at a point $p\in M$, R_p; is the same as the curvature tensor of a symmetric space (that can change with the point of p). Thus locally symmetric spaces are obviously semi-symmetric, but the converse is not true (see [4], [5]). A complete intrinsic classification of these spaces was given by Szabó (see [13]). However, it is interesting to investigate the semi-symmetry of special Riemannian manifolds. Nomizu proved that if M^{n} is a complete, connected semi-symmetric hypersurfaces of an Euclidean space $R^{n+1}, n>3$, i.e., $R\cdot R=0$, then M^{n} is locally symmetric, i.e., $\nabla R=0$. For the case of a compact Kaehlerian manifold, Ogawa proved that if it is semi-symmetric then it must be locally symmetric (see [9]).

Furthermore, the conditions $R(X,Y)\cdot P=0$, $R(X,Y)\cdot C=0$ and $R(X,Y)\cdot C=0$ are called projectively semi-symmetric, conformally (Weyl) semi-symmetric and concircularly semi-symmetric respectively, where $R(X,Y)$ is considered as derivation of tensor algebra at each point of the manifold (see [1]).

In this paper, we obtain some results on almost alpha Kenmotsu manifolds with dimension 3 satisfying some certain conditions and η-parallelity of φh. Finally, an illustrative example on three dimensional almost alpha Kenmotsu manifold depending on alpha is constructed.

2. Research Method

Almost contact manifolds have odd-dimension. Let us denote the manifold by M^{2n+1}. Then it carries two fields and a 1-form. These fields are denoted by φ and ξ. The field φ represents the endomorphisms of the tangent spaces. The field ξ is called characteristic vector field. Also, η is an 1-form given by

$$\varphi^2=-I+\eta\otimes\xi, \eta(\xi)=1,$$

such that $I:TM^{2n+1}\to TM^{2n+1}$ is the identity transformation. In light of the above information, it follows that

$$\varphi\xi=0, \eta\circ\varphi=0,$$

and the (1,1)-tensor field φ is of constant rank $2n$ (see [16]). Let $(M^{2n+1},\varphi,\xi,\eta)$ be an almost contact manifold. This manifold called normal if the following tensor field N

$$N=[\varphi,\varphi]+2d\eta\otimes\xi,$$

vanishes identically. Furthermore, $[\varphi,\varphi]$ represents the Nijenhuis tensor of the tensor field φ. It is well known that $(M^{2n+1},\varphi,\xi,\eta)$ inducts the following Riemannian metric g

$$g(\varphi X,\varphi Y)=g(X,Y)-\eta(X)\eta(Y),$$

for arbitrary vector fields X,Y on M^{2n+1}. The above metric g is said to be a compatible metric. Thus the structure given with this quadruple called almost contact metric structure. Such manifolds are said to be the same name. According to the above equation, we have $\eta=g(.,\xi)$. Moreover, the Φ represents the 2-form of the manifold that is given by

$$\Phi(X,Y)=g(\varphi X,Y),$$

Then it is called the fundamental 2-form of M^{2n+1}. For an almost contact metric manifold, if both η and Φ are closed, then it is said to be an almost cosymplectic manifold. In addition, if an almost contact metric manifold holds the following equations

$$d\eta = 0, d\Phi = 2\eta \wedge \Phi.$$

Then it is called an almost Kenmotsu manifold. These manifolds are studied in (see [5], [6] and [7]). An almost contact metric manifold M^{2n+1} is said to be almost alpha Kenmotsu if $d\eta = 0$ and $d\Phi = 2\alpha\eta \wedge \Phi$, α being a non-zero real constant.

We define $A = -\nabla\xi$ and $h = (1/2)L_\xi\varphi$ for all vector fields where alpha is a smooth function such that $d\alpha \wedge \eta = 0$ and recall that $A(\xi) = 0$ and $h(\xi) = 0$. Then we have

$$\nabla_X\xi = -\alpha\varphi^2X - \varphi hX, \tag{2.1}$$

$$(\varphi h)X + (h\varphi)X = 0, \tag{2.2}$$

$$(\varphi A)X + (A\varphi)X = -2\alpha\varphi, \quad tr(h) = 0, \tag{2.3}$$

for arbitrary vector fields X, Y on M^{2n+1}, (see [11] and [12]).

The Weyl curvature tensor is a measure of the curvature of spacetime and differs from the Riemannian curvature tensor. It is the traceless component of the Riemannian tensor which has the same symmetries as the Riemannian tensor. The most important of its special property that it is invariant under conformal changes to the metric. Namely, if $g^* = kg$ for some positive scalar functions k, then the Weyl tensor satisfies the equation $W^* = W$. In other words, it is called conformal tensor (see [16]).

Let M be a $(2n+1)$-dimensional Riemannian manifold with metric g. The Ricci operator Q of (M, g) is defined by $g(QX, Y) = S(X, Y)$, where S denotes the Ricci tensor of type (0,2) on M. Weyl constructed a generalized curvature tensor of type (1,3) on a Riemannian manifold which vanishes whenever the metric is (locally) conformally equivalent to a flat metric; for this reason he called it the conformal curvature tensor of the metric. The Weyl conformal curvature tensor is defined by

$$\begin{aligned} C(X,Y)Z \; = \; & R(X,Y)Z - (1/(2n-1))[S(Y,Z)X - S(X,Z)Y \\ & +g(Y,Z)QX - g(X,Z)QY] \\ & +(r/(2n(2n-1)))[g(Y,Z)X - g(X,Z)Y], \end{aligned} \tag{2.4}$$

for any vector fields X, Y on M, where R, r are denoting the Riemann curvature tensor and scalar curvature of M, respectively (see [16]).
Moreover, the concircular curvature tensor $\bar{C}$ and the projective curvature tensor P of (M^{2n+1}, g) are defined as

$$\bar{C}\,(X,Y)Z = R(X,Y)Z - (r/2n(2n+1))(g(Y,Z)X - g(X,Z)Y), \tag{2.5}$$

$$P(X,Y)Z = R(X,Y)Z - (1/2n)[S(Y,Z)X - S(X,Z)Y], \tag{2.6}$$

respectively, where S is the Ricci tensor, $r = tr(S)$ is the scalar curvature and $X, Y, Z \in \chi(M^{2n+1})$, $\chi(M^{2n+1})$ being the Lie algebra of vector fields of M^{2n+1} (see [16]).

Also, we have the following curvature relations on $(M^{2n+1},\varphi,\xi,\eta,g)$ almost alpha-cosymplectic manifold. Here alpha is a smooth function where $d\alpha \wedge \eta = 0$, and $l = R(.,\xi)\xi$ is the Jacobi operator (see [11] and [12]):

$$R(X,Y)\xi = (\nabla_Y \varphi h)X - (\nabla_X \varphi h)Y - \alpha[\eta(X)\varphi hY - \eta(Y)\varphi hX] + [\alpha^2 + \xi(\alpha)][\eta(X)Y - \eta(Y)X], \quad (2.7)$$

$$lX = [\alpha^2 + \xi(\alpha)]\varphi^2 X + 2\alpha\varphi hX - h^2 X + \varphi(\nabla_\xi h)X, \quad (2.8)$$

$$(\nabla_\xi h)X = -\varphi lX - [\alpha^2 + \xi(\alpha)]\varphi X - 2\alpha hX - \varphi h^2 X, \quad (2.9)$$

$$S(X,\xi) = -2n[\alpha^2 + \xi(\alpha)]\eta(X) - \big(div(\varphi h)\big)X, \quad (2.10)$$

$$S(\xi,\xi) = -\big[2n\big(\alpha^2 + \xi(\alpha)\big) + tr(h^2)\big]. \quad (2.11)$$

Now, we give some necessary concepts for later usage:

Proposition 2.1. Let $(M^{2n+1},\varphi,\xi,\eta,g)$ be an almost alpha Kenmotsu manifold. If the tensor field φh is η-parallel, then we have

$$(\nabla_X \varphi h)Y = \eta(X)[lY - (\alpha^2 + \xi(\alpha))\varphi^2 Y - 2\alpha\varphi hY + h^2 Y] - \eta(Y)[\alpha\varphi hX - h^2 X] - g(Y, \alpha\varphi hX - h^2 X)\xi, \quad (2.12)$$

for all vector fields X, Y on M^{2n+1} (see [11] and [12]).

Proposition 2.2. An almost alpha Kenmotsu manifold with η-parallel tensor φh satisfies the following relation

$$R(X,Y)\xi = \eta(Y)lX - \eta(X)lY, \quad (2.13)$$

where $l = R(.,\xi)\xi$ is the Jacobi operator with respect to the characteristic vector field ξ (see [11] and [12]).

Definition 2.1. Any symmetric (1,1)-type tensor field E on a Riemannian manifold (M,g) is said to be a η-parallel tensor if it satisfies the relation

$$g\big((\nabla_{X^T} E)Y^T, Z^T\big) = 0, \quad (2.14)$$

for all tangent vectors X^T orthogonal to ξ where X defined by $X = X^T + \eta(X)\xi$ is tangentially part of X and $\eta(X)\xi$ the normal part of X (see [2]).

3. Results and Analysis

In this section, we study some certain tensor fields on three dimensional almost alpha Kenmotsu manifolds. Thus we state the following results:

Theorem 3.1. Let M be an almost alpha Kenmotsu manifold with dimension 3. If it is projectively flat then it is a manifold of scalar curvature

$$r = 3S(\xi,\xi) + 2\, tr(\varphi(\nabla\xi h)), \quad (3.1)$$

where α is parallel along the vector field ξ.

Proof. It can be proved by a similar technique in [9] for 3-dimensional almost alpha Kenmotsu manifold. By the help of $\xi(\alpha) = 0$, the proof is obvious.

Theorem 3.2. If M be a semi symmetric three dimensional almost alpha Kenmotsu manifold with φh Codazzi condition, then there exists no alpha Kenmotsu structure where α is parallel along the vector field ξ.

Proof. Suppose that φh satisfies the Codazzi condition, i.e., this is essentially same as

$$g((\nabla Y\varphi h)X, Z) - g((\nabla X\varphi h)Y, Z) = 0, \tag{3.2}$$

Follows from (2.7), we have

$$R(X,Y)\xi = \alpha^2[\eta(X)Y - \eta(Y)X] - \alpha[\eta(X)\varphi hY - \eta(Y)\varphi hX] \tag{3.3}$$

where α is parallel along the vector field ξ.

Then considering the condition $R(X,Y) \cdot R$ which is defined as

$$\begin{aligned}(R(X,Y)R)(U,V)W &= R(X,Y) \cdot R(U,V)W - R(R(X,Y)U,V)W \\ &\quad -R(U,R(X,Y)V)W - R(U,V)R(X,Y)W,\end{aligned} \tag{3.4}$$

As it has been supposed that $R(X,Y) \cdot R = 0$, thus we have

$$\begin{aligned}0 &= R(X,\xi) \cdot R(U,V)\,\xi - R(R(X,\xi)U,V)\,\xi \\ &\quad -R(U,R(X,\xi)V)\,\xi - R(U,V)R(X,\xi)\xi.\end{aligned} \tag{3.5}$$

where $R(X,Y) \cdot R = 0$ is equivalent to $R(X,\xi) \cdot R = 0$.

Here, we must find the four statements for the right side of (3.5) separately.

From (3.3) and (3.5), (3.5) reduces to

$$\begin{aligned}0 &= \alpha^3[\eta(U)g(hV,\varphi X) - \eta(V)g(hU,\varphi X)] \\ &\quad + \alpha^2[\,\eta(U)g(hX,hV) - \eta(V)g(hU,hX)]\end{aligned} \tag{3.6}$$

for arbitrary vector fields on M. Then in view of (3.6), we obtain

$$\alpha[\eta(U)g(hV,\varphi X) - \eta(V)g(hU,\varphi X)] = -\eta(U)g(hX,hV) + \eta(V)g(hU,hX). \tag{3.7}$$

It follows that

$$0 = \eta(U)g(h^2X,V) - \eta(V)g(h^2X,U) \tag{3.8}$$

for the case $\alpha = 0$. Thus this contradicts to our assumption. Hence, the tensor field h does not have to vanish. So it completes the proof.

Theorem 3.3. *Let* $(M^{2n+1},\varphi,\xi,\eta,g)$ be a three dimensional concircularly flat almost alpha Kenmotsu manifold. If it satisfies the η-parallelity condition of φh, then it is a concircular flat manifold for vector fields on the distribution D.

Proof. Using the definition of concircular curvature tensor $\bar{C}$ and making use of η-paralletity condition of φh, we have

$$\eta(\bar{C}\,(X,Y)Z) = -\eta(Y)g(lX,Z) + \eta(X)g(lY,Z)$$
$$-(r/6)(\eta(X)g(Y,Z) - \eta(Y)g(X,Z)). \quad (3.9)$$

Taking $Z = \xi$ in (3.9) we get

$$\eta(\bar{C}\,(X,Y)\xi) = 0. \quad (3.10)$$

Again taking $X = \xi$ in (3.10) we have

$$\eta(\bar{C}(\xi,Y)Z) = g(lY,Z) - (r/6)(g(Y,Z) - \eta(Y)\eta(Z)). \quad (3.11)$$

Now, we consider the condition $R(X,Y)\cdot\bar{C}$ which is defined by

$$(R(X,Y)\bar{C})(U,V)Z = R(X,Y)\cdot\bar{C}(U,V)Z - \bar{C}(R(X,Y)U,V)Z$$
$$-\bar{C}(U,R(X,Y)V)Z - \bar{C}(U,V)R(X,Y)Z. \quad (3.12)$$

Making use of (2.5), (2.7) and (2.13) we obtain

$$g(lY,Z) = (r/6)[g(Y,Z) - \eta(Y)\eta(Z)] \quad (3.13)$$

Let $R(X,Y)\cdot\bar{C} = 0$, then we have

$$0 = R(X,Y)\cdot\bar{C}(U,V)Z - \bar{C}(R(X,Y)U,V)Z$$
$$-\bar{C}(U,R(X,Y)V)Z - \bar{C}(U,V)R(X,Y)Z. \quad (3.14)$$

where $\bar{C}(U,V,Z,Y) = g(\bar{C}(U,V)Z,Y)$.

Now, let us calculate the right side of (3.14). By simple direct calculations, we get

$$\bar{C}(U,V,lX,Z) = -K[\eta(U)\eta(V)g(lX,Z) + \eta(Z)\eta(V)g(lX,U) + \eta(U)\eta(Z)g(lX,V)]$$
$$-\eta(Z)\eta(X)g(lX,V) + \eta(Z)\eta(X)\eta(V). \quad (3.15)$$

where $K = -\eta(U)\eta(V)(\frac{r}{6})$.
Also, we denote by D the distribution orthogonal to ξ, i.e., $D = ker(\eta) = \{X\!:\eta(X) = 0\}$.

For $X, Y \in D$, (3.15) leads to $\bar{C} = 0$. So it completes the proof.

Example 3.1. Let us denote the standart coordinates of $R^3(x,y,z)$ and consider 3-dimensional manifold $M \subset R^3$ defined by $M = \{(x,y,z) \in R^3\!:\; z \neq 0\}$. The vector fields are

$$e_{1=}e^{z^3}\left(\frac{\partial}{\partial x}\right), e_{2=}e^{z^3}\left(\frac{\partial}{\partial y}\right), e_3 = \left(\frac{\partial}{\partial x}\right),$$

It is clear that $\{e_1, e_2, e_3\}$ are linearly independent at each point of M. Let g be the Riemannian metric defined by

$$g(e_1,e_1) = g(e_2,e_2) = g(e_3,e_3) = 1,\;\; g(e_1,e_2) = g(e_1,e_3) = g(e_2,e_3) = 0,$$

and given by the tensor product

$$g = (\frac{1}{e^{2z^3}})(dx \otimes dx + dy \otimes dy) + dz \otimes dz.$$

Let η be the 1-form defined by $\eta(X) = g(X, e_3)$ for any vector field X on M and φ be the (1,1) tensor field defined by $\varphi(e_1) = e_2, \varphi(e_2) = -e,\ \varphi(e_3) = 0$. Let h be the (1,1) tensor field defined by $h(e_1) = -\lambda e_1, h(e_2) = \lambda e_2$ and $h(e_3) = 0$. Then using linearity of g and φ, we have

$$\varphi^2 X = -X + \eta(X)e_3,\ \ \eta(e_3) = 1,\ \ g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y),$$

for any vector fields on M.

It follows that the structure of (φ, ξ, η, g) can easily be obtained. So it is sufficient to check that the only non-zero components of the second fundamental form Φ are

$$\Phi((\partial/(\partial x)), (\partial/(\partial y))) = -\Phi((\partial/(\partial y)), (\partial/(\partial x))) = -(\frac{1}{e^{2z^3}})$$

and hence

$$\Phi = -(\frac{1}{e^{2z^3}})(dx \wedge dy), \tag{3.16}$$

where $\Phi(e_{1,}e_2)$=-1 and otherwise $\Phi(e_{i,}e_j)$=-1 for $i \leq j$. Thus the exterior derivation of Φ is given by

$$d\Phi = 6z^2 e^{-2z^3}(dx \wedge dy \wedge dz). \tag{3.17}$$

Since $\eta = dz$, by the help of (3.16) and (3.17) we have

$$d\Phi = -6z^2(\eta \wedge \Phi),$$

where α defined $\alpha(z) = -3z^2$. Hence, the manifold is an alpha Kenmotsu.

4. Conclusion

Semi symmetric and locally symmetric spaces are important characterization for Riemannian manifolds. We know that locally symmetric spaces are obviously semi-symmetric, but the converse is not true (see [4] and [5]). In this paper, we study almost alpha-cosymplectic manifolds in the light of (1.1) and (2.13). Some certain results are obtained related to some semi symmetric conditions on three dimensional almost alpha Kenmotsu manifolds where alpha is a smooth function such that $d\alpha \wedge \eta = 0$.

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